A SURFACE SOURCE SINGULARITY MODEL FOR TIME AVERAGED FLOW AROUND CYLINDRICAL STRUCTURES

A. G. T. Sugathapala
Dept. Mechanical Engineering, University of Moratuwa.
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ABSTRACT

A simple method of solution to the problem of two-dimensional separated flow around multiple-body circular cylindrical structures is presented. The method is applied to the case of two identical cylinders where the free-stream direction is perpendicular to the line joining the centers. The effect of the bodies and their wakes on the outer potential flow is modelled by surface source distributions over the front wetted surfaces. The flow in the wakes, generated by the source distributions, is ignored. The condition of the zero normal velocity is applied over the front surface to obtain an integral equation for the source strength. This equation is solved with the condition of finite velocities at the separation points to find the unknown source strength distribution and the positions of the separation points. The pressure distribution on the front wetted surface is then calculated through Bernoulli's equation. The pressure distribution on the rear surface exposed to the wake is assumed to be determined by the separation values. The effects of the distance between the bodies on the flow pattern, position of the separation points, wake under-pressure and the resultant forces are analysed. The most significant feature of the present flow model is that no empirical parameters are needed for the analysis. The empirical parameters used in other models, the back pressure coefficient and the positions of the separation points, can be predicted through the present model.

INTRODUCTION

Analysis of flow around and forces on bluff bodies is an important applied fluid mechanics problem. Aerodynamics of buildings and structures, hydrodynamics of under-water structures, flow around heat exchanger tube arrays are few examples. The flow around a bluff body at high Reynolds numbers is characterized by extensive boundary layer separation and formation of a broad wake behind the body. The boundary of the separated flow (so-called free shear layer) is thin and usually quite steady within a certain distance down stream of the separation points. The pressure in this near-wake region is approximately constant and lower than the free stream pressure. The body surface exposed to the wake is subjected to this uniform wake under-pressure. Further downstream the separated shear layers become unstable and roll up to form large scale eddies (vortices). This far-wake region is unsteady and highly turbulent. Therefore, even for steady on-coming flow, the resultant flow around a bluff body is unsteady. The body is then acted upon by time-dependent pressure loads. The mean forces are of particular interest and can be determined by averaging over a time much larger than the characteristic period of the unsteady flow (the vortex shedding periods). As a result of the wake under-pressure, the time-averaged mean forces acting on bluff bodies are relatively high and are mainly due to the non-symmetrical pressure distribution. At high Reynolds numbers contribution of the viscous forces is negligible.

Considerable progress has been made in the analysis of the flow past bluff bodies through the numerical solution of the complete Navier-Stokes equations. However results are restricted to laminar flow at low Reynolds numbers. Extensions to higher Reynolds numbers involve excessive computing time and suffers from uncertainties in turbulent modelling. A simple alternative approach is to utilize an inviscid flow
model that represents the important characteristics of the real flow. In fact the outer-
flow region as well as the near-wake region could be considered as inviscid and
irrotational (potential) and it is possible to model the outer flow and the pressure
distribution on the body accurately by potential flow models. However, all such
models involve empirical parameters, such as positions of the separation points and
back pressure coefficient (which determines the wake under-pressure), since the
separation of flow and formation of wake are governed by the viscous properties of
the fluid and can not be expected to be modelled completely by inviscid
approximations. The existing models for time-averaged mean flow around bluff
bodies are briefly reviewed in the next section.

Almost all the existing bluff body potential flow models are limited to the case of
flow around an isolated body. However in most of the practical applications fluid
flow takes place around multiple bluff bodies (or a body near a solid wall), where the
interaction between them are important. Another important application is model
testing in wind tunnels where the flow takes place within a confined space. Analysis
of such a flow problem is difficult since the phenomenon of flow separation and the
interaction of bodies and wakes are very complicated.

The main purpose of this paper is to introduce a relatively simple potential flow
model which can be applied to treat the flow around multiple-body circular cylindrical
structures where the bodies are not in the wake regions of the others. A simple
example of such a problem is that of the flow about two identical cylinders. In the
case considered, the free stream direction is assumed to be perpendicular to the line
joining the centers. The proposed method is an extension of the surface source
singularity model presented by Hess (1973) for the case of flow past a circular
cylinder. The effect of bodies and their wakes on the outer-flow is modelled by
source distributions on the front wetted surfaces. The source strength distribution and
the position of the separation points are determined by appropriate boundary and flow
conditions, for a given distance between the centers. Prediction of the flow pattern,
the pressure and velocity distributions are then straightforward. No empirical
parameters are needed for the model and both positions of the separation points and
back pressure coefficients can be predicted.

**POTENTIAL FLOW MODELS**

The salient features of uniform flow around long bodies of constant cross-sections
suggests the possibility of modelling the time-averaged flow quantities such as
velocity and surface pressure distribution using two-dimensional and incompressible
potential flow models. These models basically fall into the following three categories:

(a) free-streamline models using conformal mapping,
(b) surface singularity models,
(c) free-streamline models with surface and wake singularities.

In each of these models back pressure coefficient is introduced as a free parameter
(as well as the position of the separation points in the case of continuously curved
body) in order to account for the essential features of a complicated process of
viscous dissipation in the wake, and to replace the real wake flow by a simplified
model within the framework of the potential theory. These flow models do not provide a good description of the far-wake region at all, and the validity will have to be justified by their agreement with experimental observation of the actual flow field near the body.

In the models under category (a), the free-streamlines (or surface of discontinuity in velocity) are taken as the idealization of the separating shear layers. In conventional models the shape of these lines are determined by a series of conformal transformations of complex velocity and complex potential planes to the physical plane, and thereby velocity and pressure distributions in the outer-flow regime are calculated (Wu 1972). More simplified modelling method was presented by Parkinson and Jandali (1970), where a circular cylinder in a transformation plane was used instead of complex velocity and complex potential planes. Calculation of the flow field around the circle is simple and the conformal transformation of the circle to the body shape in the physical plane results in the required solution. This method has been extended and applied successfully for the flow around a number of bluff body shapes (Yeung & Parkinson 1997). In general, all these methods are limited to two-dimensional flow around a few standard bluff body shapes (such as circular cylinder, flat plate) because the conformal mapping technique depends on knowing suitable transformations.

In the method of surface singularity modelling, the body surface is replaced by a distribution of flow singularities (sources or vortices). The strength of the singularity varies over the surface such that at every point on the boundary a prescribed boundary condition is satisfied. Comprehensive reviews of these methods were presented by Hess & Smith (1966) on surface source models and by Lewis (1981) on surface vortex models. A large class of solutions for potential flow around bodies (including multiple-body configurations) can be generated by this indirect method of solution. However, relatively few works deal with the problem of separated flow past bluff bodies. One such method is the model presented by Hess (1973), where only the front wetted surface of the body is replaced by a continuous source distribution. The flow from the source distribution models the effect of the wake on the outer potential flow. The most significant result of this model is that it gives the separation angle for the circular cylinder at 77.45° from the front stagnation point (which is close to the separation point for laminar boundary layer flow) and a back pressure coefficient of -0.96. In all the other models these two quantities are taken as free parameters. This model was extended by the present author by adding a down-stream sink to close the wake (Sugathapala 1996). Different positions of the sink correspond to different back pressure coefficients, thus permitting a wide range of solutions. The main contribution of this closed wake model is its application to unsteady flow. An open wake model gives infinite potential at the far field in the case of unsteady flow, which is physically inadmissible.

Free-streamline models with surface and wake singularities assume that the vorticity in the wake is concentrated in relatively thin shear layer so that the rotational wake can be modelled by pair of vortex sheets. The body surface too is replaced by a vortex (or source) distribution. The position of the vortex sheets (wake shape) is calculated by an iterative procedure. It is possible to extend this method to incorporate the influence of boundary layer displacement effects, viscous diffusion
and dissipation of the wake vorticity and other important features in the real wake (Mukherjea & Bandyopadhyay 1990).

SURFACE SINGULARITY MODEL FOR FLOW AROUND TWO CYLINDERS

Formulation of the Problem

Consider two-dimensional, incompressible flow, uniform at infinity, past two identical circular cylinders in parallel. The free-stream direction is assumed to be perpendicular to the plane of the axes of the cylinders. Let the radius of the cylinders be \( R \), distance between centers be \( H \) and the free-stream velocity be \( U \).

\[ \begin{align*}
\text{Figure (1): Two Identical Cylinders in a Uniform flow. }
\end{align*} \]

The flow configuration and the selected coordinate axis system are shown in Figure (1). The effect of the bodies on the on-coming flow is modelled by source distributions over the front surfaces \( S_1 \) and \( S_2 \) where the flow is attached. Flow from each source distribution occupies a semi-infinite region of finite width behind the corresponding body, which represents the shape of the wake. The separating shear layers in the real flow are assumed to be represented by the boundary of the source flow (free streamlines) and the flow inside the wake is ignored. The resulting flow is symmetrical about the \( X \)-axis, hence the source distributions on the two cylinders are identical. The strength of the source distribution \( Q(\theta) \) per unit length and the positions of the separation points \( \beta_1 \) and \( \beta_2 \) can be
determined by the Neumann boundary condition of zero normal velocity on the surface of a cylinder and a condition that the tangential velocities are finite at the separation points.

![Diagram](image)

**Figure (2): Surface Source Distributions on Cylinders.**

The velocities at the surface point $P$, induced by the source elements at the point $T_1$ on the same cylinder and the point $T_2$ on the other cylinder are given by the expressions

$$\delta V_1 = \frac{RQ(\phi) \, d\phi}{4\pi} \frac{T_1 P}{T_1 P^2}$$

and

$$\delta V_2 = \frac{RQ(\phi) \, d\phi}{4\pi} \frac{T_2 P}{T_2 P^2}$$

where,

$$T_1 P = R(\cos \phi - \cos \theta) e + R(\sin \phi - \sin \theta) j$$

and

$$T_2 P = R(\cos \phi - \cos \theta) e + (H + R(\sin \phi + \sin \theta)) j$$

with respect to the coordinate axis system $x$-$y$, $e$ and $j$ are the corresponding unit vectors and $-\beta_2 \leq \phi \leq \beta_1$, $-\beta_1 \leq \theta \leq \beta_2$ (see Figure (2)). The unit normal outward from the cylinder and the tangential vectors at $P$ are

$$n = -\cos \theta e + \sin \theta j \quad \text{and} \quad t = \sin \theta e + \cos \theta j,$$

respectively.

Hence the normal and tangential velocities induced by the source elements are given by

$$\delta V_n = n \cdot \delta V_1 + n \cdot \delta V_2 \quad \text{and} \quad \delta V_t = t \cdot \delta V_1 + t \cdot \delta V_2,$$

respectively.

After substituting above expressions, the resultant normal and tangential velocities at the point $P$ due to the source distributions on the two cylinders can be evaluated by
integrating over the front wetted surfaces, as

\[ V_{Q_1}(\theta) = \frac{Q(\theta)}{2} + \frac{1}{4\pi} \int_{-\beta_2}^{\beta_1} Q(\phi) d\phi + \frac{1}{4\pi} \int_{-\beta_2}^{\beta_1} f_1(\phi,\theta,e)Q(\phi) d\phi \] (1)

and

\[ V_{Q_2}(\theta) = \frac{1}{4\pi} \int_{-\beta_2}^{\beta_1} \frac{\sin(\theta - \phi)}{1 - \cos(\theta - \phi)} Q(\phi) d\phi + \frac{1}{4\pi} \int_{-\beta_2}^{\beta_1} f_2(\phi,\theta,e)Q(\phi) d\phi \] (2)

after some algebraic simplifications, where

\[ f_1(\phi,\theta,e) = \frac{1 - \cos(\theta + \phi) + 2e \sin \theta}{1 - \cos(\theta + \phi) + 2e(\sin \theta + \sin \phi) + 2e^2} \] (3)

\[ f_2(\phi,\theta,e) = \frac{2e \cos \theta + \sin(\theta + \phi)}{1 - \cos(\theta + \phi) + 2e(\sin \theta + \sin \phi) + 2e^2} \] (4)

with \( e = H/2R > 1 \).

The origin of the term \( Q(\theta)/2 \) in expression (1) for \( V_{Q_1}(\theta) \) is due to the delta function that is brought by the limiting process as \( \phi \to \theta \). The corresponding contribution on \( V_{Q_2}(\theta) \) is zero and does not arise in expression (2). The normal and tangential velocity components at the point \( P \) due to the on-coming uniform flow are \(-U \cos \theta \) and \( U \sin \theta \) respectively. Then the application of Neumann boundary condition of zero normal velocity on the cylinder surface leads to

\[ 2\pi \sigma(\theta) + \int_{-\beta_2}^{\beta_1} \sigma(\phi) d\phi + \int_{-\beta_2}^{\beta_1} f_1(\phi,\theta,e) \sigma(\phi) d\phi = \cos \theta, \] (5)

where \( \sigma(\theta) = Q(\theta)/4\pi U \) is the non-dimensional source distribution. This is a Fredholm integral equation of the second kind for the source distribution \( \sigma(\theta) \). In general there is no difficulty of solving this equation for given \( \beta_1 \), \( \beta_2 \) and \( e \), but a complete analytical solution is not possible since the kernel of the second integral depends on both \( \phi \) and \( \theta \), and is not a simple function. Furthermore, a direct numerical solution too is not possible since the limits of the integration \( \beta_1 \) and \( \beta_2 \) (separation angles) are unknowns and have to be determined by some other considerations. It can be shown that at the separation points as \( \theta \to \beta_1 \) and \( \theta \to \beta_2 \), \( V_{Q_1}(\theta) \to \infty \) (see equation (2)), which is physically inadmissible, unless

\[ \sigma(\beta_1) = 0 \quad \text{and} \quad \sigma(-\beta_2) = 0, \] (6)

respectively. Substitution of these expressions into equation (5) leads to the expressions required to find the location of separation points, as

\[ \cos \beta_1 = \int_{-\beta_2}^{\beta_2} [1 + f_1(\beta_1,\phi,e)] \sigma(\phi) d\phi \] (7)

and

\[ \cos \beta_2 = \int_{-\beta_2}^{\beta_2} [1 + f_1(\beta_2,\phi,e)] \sigma(\phi) d\phi. \] (8)

Now the complex fluid mechanics problem has reduced to a relatively simple mathematical problem of solving equations (5), (7) and (8) for \( \sigma(\theta), \beta_1 \) and \( \beta_2 \), together with the conditions given in the equation (6), where \(-\beta_2 \leq \theta \leq \beta_1 \). Once these equations have been solved, calculations of the velocity and pressure field in the outer flow regime, pressure distribution on the wetted surface and the streamline patterns are straightforward.
Numerical Method of Solution

In the present analysis the solution of equation (5) is obtained numerically by the selection of a sufficient number of equi-spaced pivotal points distributed over the front wetted surface. The continuous source distribution is then replaced by discrete sources of unknown strengths at each point. Now trapezoidal integration is introduced to replace the integral equation (5) by a set of simultaneous equations. Let the number of pivotal points be \( N+1 \) including the separation points \( S_1 \) and \( S_2 \). Then equation (5) applied to \( i^{th} \) pivotal point becomes

\[
2\pi \sigma_i + \left( \frac{\beta_1 + \beta_2}{N} \right) \sum_{j \neq i}^{N-1} \sigma_j + \left( \frac{\beta_1 + \beta_2}{N} \right) \sum_{j=1}^{N-1} f_1(\phi_j, \theta_i, \epsilon) \sigma_j = \cos \theta_i, \tag{9}
\]

where \( i = 1, 2, \ldots, N-1 \), \( \sigma_i = \sigma(\theta_i) \), \( \theta_i = -\beta_2 + (\beta_1 + \beta_2)i/N \), \( \phi_j = -\beta_2 + (\beta_1 + \beta_2)j/N \) with \( \sigma_0 = \sigma(-\beta_2) = 0 \) and \( \sigma_N = \sigma(\beta_2) = 0 \) (see equation (6)). Application of above equation to the separation points at \( i = 0 \) and \( i = N \) leads to the conditions required to find the positions of the separation points as

\[
\cos \beta_1 = \left( \frac{\beta_1 + \beta_2}{N} \right) \sum_{j=1}^{N-1} [1 + f_1(\phi_j, \beta_1, \epsilon)] \sigma_j \tag{10}
\]

and

\[
\cos \beta_2 = \left( \frac{\beta_1 + \beta_2}{N} \right) \sum_{j=1}^{N-1} [1 + f_1(\phi_j, -\beta_2, \epsilon)] \sigma_j. \tag{11}
\]

Note that these equations are equivalent to equations (7) and (8), respectively. Equation (9) represents \( N-1 \) number of linear equations for \( N-1 \) number of unknowns \( \sigma_1, \sigma_2, \ldots, \sigma_{N-1} \) in the matrix form

\[
[A_{ij}][\sigma_j] = [K_j], \tag{12}
\]

where \( i, j = 1, 2, \ldots, N-1 \).

An iteration procedure is employed to solve equations (10), (11) and (12). For a given value of \( \epsilon \), equation (12) is solved for \( \sigma_j \) based on the method of LU factorization with assumed values for \( \beta_1 \) and \( \beta_2 \). Substitution of \( \sigma_j \) in equations (10) and (11) gives better approximations for \( \beta_1 \) and \( \beta_2 \). This procedure is performed till the solutions converge to the correct values with required accuracy. Since equation (5) is a Fredholm integral equation of the second kind, the diagonal elements of the coefficient matrix \([A_{ij}]\) are predominant. As a result the efficiency of the iteration process is high and has a rapid convergence. As an example, for almost entire range of values of \( \epsilon \), the solutions converge in less than 10 iterations when \( N=500 \) with 0.001% accuracy. Only when \( \epsilon \) is very close to 1.0 (approximately \( \epsilon \approx 1.1 \)) the solution does not converge within 100 iterations. Anyway in most of the practical applications \( \epsilon \) is not very close to 1.0.

The change in the positions of the separation points with \( \epsilon \) is presented in Figure (3). It can be seen that as \( \epsilon \) decreases, \( \beta_1 \) decreases and \( \beta_2 \) increases which result in an
angular shift of the near-wake regions away from each other. When $e \gg 1$, both $\beta_1$ and $\beta_2$ approaches to $77.45^\circ$, the value corresponds to the isolated circular cylinder. The source distribution for the case $e = 1.5$ is given in Figure (4).

**Figure (3) :** Position of the Separation Points: (a)-Approximate Analytical Solution; (b)-Numerical Solution.

**Figure (4) :** Source Distribution on the Cylinder.
Approximate Analytical Solution

In the early stages of the present work, the author attempted to solve the integral equation (5) analytically, but failed to obtain a complete solution. However an approximate solution is possible, which closely agrees with the numerical result, especially when \( e > 2 \). This analytical work is reproduced in this section, as it will confirm the validity of the numerical results. In addition, the analysis will give a better insight into the problem and such analytical work will be useful as an academic exercise.

The main difficulty of obtaining an analytical solution to the integral equation (5) is the functional form of the kernel \( f_1(\phi, \theta, e) \) of the second integral, given in equation (3). There the parameter \( e = H/2R > 1 \) and, in general, \( f_1(\phi, \theta, e) \) is of order \( \delta = 1/e < 1 \). Hence one possible alternative approach in attempting an approximate solution is to expand \( f_1(\phi, \theta, e) \) as a power series of \( \delta \) and keep only the significant order terms, as

\[
f_1(\phi, \theta, \delta) = \frac{2\delta \sin \theta + \delta^2 [1 - \cos(\theta + \phi)]}{2 + 2\delta \sin \theta + \sin \phi + \delta^2 [1 - \cos(\theta + \phi)]} = \delta \sin \theta + \frac{\delta^2}{2} \cos(2\theta) - \cos(\theta - \phi) + O(\delta^3). \tag{13}
\]

The source distribution too may be expanded as a power series of \( \delta \) in the form

\[
\sigma(\theta) = \sigma_0(\theta) + \delta \sigma_1(\theta) + \delta^2 \sigma_2(\theta) + O(\delta^3). \tag{14}
\]

Substitution of expansions (13) and (14) in equation (5) leads to the following integral equations

\[
2\pi \sigma_0(\theta) + \int_{-\beta_1}^{\beta_1} \sigma_0(\phi) d\phi = \cos \theta \tag{15}
\]

\[
2\pi \sigma_1(\theta) + \int_{-\beta_1}^{\beta_1} \sigma_1(\phi) d\phi = -\sin \theta \int_{-\beta_1}^{\beta_1} \sigma_0(\phi) d\phi \tag{16}
\]

\[
2\pi \sigma_2(\theta) + \int_{-\beta_1}^{\beta_1} \sigma_2(\phi) d\phi = -\sin \theta \int_{-\beta_1}^{\beta_1} \sigma_1(\phi) d\phi + \frac{1}{2} \int_{-\beta_1}^{\beta_1} [\cos(\theta - \phi) - \cos(2\theta)] \sigma_0(\phi) d\phi \tag{17}
\]

correspond to the terms of \( O(1) \), \( O(\delta) \) and \( O(\delta^2) \), respectively.

Each of these equations is a Fredholm integral equation of the second kind and is of the simplest form. The solution to equation (15) is direct and given by

\[
\sigma_0(\theta) = \frac{(\cos \theta - c_0)}{2\pi}, \tag{18}
\]

where \( c_0 = (\sin \beta_1 + \sin \beta_2)/(2\pi + \beta_1 + \beta_2) \). Equation (16) is solved after substituting for \( \sigma_0(\theta) \) from (18), and the result is

\[
\sigma_1(\theta) = \frac{(-c_1 + c_0 \sin \theta)}{2\pi}, \tag{19}
\]

where \( c_1 = c_0(\cos \beta_1 - \cos \beta_2)/(2\pi + \beta_1 + \beta_2) \).
Similarly solution to equation (17) can be obtained, but in a more complicated form, as

$$\sigma_\delta(\theta) = a_0 + a_1\cos\theta + a_2\cos(2\theta) + b_1\sin\theta,$$  \hspace{1cm} (20)

where $a_0$, $a_1$, $a_2$, and $b_1$ are some known functions of $\beta_1$ and $\beta_2$. Same procedure may be adopted to derive expressions for other higher order terms but they are getting more and more complicated in form. Now substitution of above expressions in (14) gives the source distribution as a function of $\beta_1$, $\beta_2$, and $\delta$. Then $\beta_1$ and $\beta_2$ can be calculated by the use of the conditions stated in expression (6) as

$$\cos\beta_1 = c_0 + \delta(c_1 + c_2\sin\beta_1) - \delta^2[a_0 + a_1\cos\beta_1 + a_2\cos(2\beta_1) + b_1\sin\beta_1],$$  \hspace{1cm} (21)

and

$$\cos\beta_2 = c_0 - \delta(c_1 - c_2\sin\beta_2) - \delta^2[a_0 + a_1\cos\beta_2 + a_2\cos(2\beta_2) + b_1\sin\beta_2],$$  \hspace{1cm} (22)

up to order $\delta^2$.

A simple iterative process is adopted to find the solutions to equations (21) and (22) for $\beta_1$ and $\beta_2$, at a given value of $\delta$ and the results are presented in Figure (3). The comparison with the numerical results shows that the approximate analytical method gives satisfactory results except in the range $\delta > 0.5$ (i.e. $e < 2$) in which other higher order terms are needed for more accurate results.

Note that for the corresponding flow around an isolated cylinder, $\delta = 0$ (i.e. $e \to \infty$) equations (21) and (22) simplify to $\tan\beta = \beta + \pi$, where $\beta = \beta_1 = \beta_2$, for which the solution is $\beta = 77.45^\circ$. The corresponding source distribution is given by $\sigma(\theta) = (\cos\theta - \cos\beta)/2\pi$ and is plotted in Figure (4).

SURFACE PRESSURE DISTRIBUTION AND FORCES

The pressure distribution in the flow external to the bodies and their wakes as well as on the front wetted surface of the bodies follow from the knowledge of the velocity field, given by the expression

$$C_p = \frac{P - P_0}{\frac{1}{2}\rho U^2} = 1 - \left(\frac{V}{U}\right)^2,$$  \hspace{1cm} (23)

where $C_p$ is the pressure coefficient, $P_0$ is the free-stream pressure and $V$ is the magnitude of the velocity at the point of consideration. Both uniform flow and the source distributions contribute to the velocity $V$ and can be calculated for a known source distribution $\sigma(\theta)$. On the wetted surface of the cylinder $V = V_\phi(\theta) = U\sin\theta + V_0(\theta)$, where $-\beta_2 \leq \theta \leq \beta_1$ and $V_0(\theta)$, given in expression (2), is the contribution of the source distributions. Substitution of this expression in (23) leads to the surface pressure distribution as

$$C_p(\theta) = 1 - \left[\frac{\beta_2}{\beta_1} \int_{-\beta_2}^{\beta_1} \left(\frac{\sin(\theta - \phi)}{1 - \cos(\theta - \phi)} + f_2(\phi, \theta, e)\right)\sigma(\phi)d\phi\right].$$  \hspace{1cm} (24)

Now the pressure coefficient at the separation points $S_1$ and $S_2$, which determine the pressure distribution in the near-wake, are given by $C_{p\phi_1} = C_p(\beta_1)$ and $C_{p\phi_2} = C_p(-\beta_2)$, respectively. Variation of these quantities with $e$ is given in Figure (5).
Figure (5): Variation of Back Pressure Coefficients.

For an isolated cylinder \((e \rightarrow \infty)\) \(C_{ph1} = C_{ph2} = -0.95\), and for case of two cylinders both \(C_{ph1}\) and \(C_{ph2}\) are lower, resulting higher forces. The change in \(C_{ph1}\) is not very significant but \(C_{ph2}\) shows very high suction (between the cylinders) as \(e \rightarrow 1\). Since \(C_{ph1} \neq C_{ph2}\), the pressure in the near-wake is not uniform which gives rise to some difficulty in predicting pressure loading on the rear surface of the cylinder exposed to the wake. It cannot be expected to predict this variation through potential flow modelling and some empirical information is needed for further calculations, especially to predict the forces. In the present analysis the back pressure coefficient \(C_{ph}(\theta)\) is assumed to be varied linearly from \(C_{ph1}\) at \(\theta = \beta_1\) to \(C_{ph2}\) at \(\theta = -\beta_2\). For an example, pressure distributions on the cylinder for the case \(e = 1.5\) and for the case of an isolated cylinder are presented in Figure (6).

Figure (6): Pressure Distribution on the Cylinder.
With the above assumption for the back pressure coefficients, the drag coefficient $C_D$ and the lift coefficient $C_L$ may be calculated by integrating the pressure distribution over the body surface, as

$$C_D = \frac{D}{\frac{1}{2} \rho U^2(2R)} = \frac{1}{2} \int \cos \theta C_p(\theta) d\theta$$

and

$$C_L = \frac{L}{\frac{1}{2} \rho U^2(2R)} = \frac{1}{2} \int \sin \theta C_p(\theta) d\theta,$$

where $D$ and $L$ are the drag and lift forces per unit length in positive $x$ and $y$ directions, respectively. The predicted force coefficients for different center distances are given in Figure (7).

![Figure (7): Variation of Force Coefficients.](image)

For an isolated cylinder $C_D = 1.06$ and $C_L = 0$. It can be seen from the Figure (7) the effect of the adjacent body is to increase the drag force and to generate a lift force. For most of center distances the lift force is small and positive (i.e. in a direction away from the adjacent body). However, for closely spaced bodies (approximately when $\epsilon < 1.5$), the lift force becomes negative and relatively high. This behaviour can be explained with the aid of flow pattern and the surface pressure distribution. One effect of the adjacent body is the angular shift of the near-wake, away from that body, exposing higher top surface area to the wake. This results in positive lift force. The other effect is to accelerate the flow between the cylinders thereby creating high suction (low pressure). This also makes the wake pressure in the bottom region to be lower than the top region, resulting negative effect on the lift force. Therefore the direction of the lift force is determined by the relative magnitudes of these two contributions.
PATTERN OF FLOW

Kinematic aspects of the flow (streamline pattern) is also important and will give a better qualitative understanding of the problem. Consider a point $P(x,y)$ in the outer flow field. The position of $P$ relative to the source elements at $T_1$ and $T_2$ are represented in polar coordinates by $(r_1, \theta_1)$ and $(r_2, \theta_2)$, as shown in Figure (8).

Figure (8) : Calculation of Streamline Pattern.

The complex potential function for the flow field is

$$F(z) = U z + \frac{R}{2\pi} \int_{\theta_1}^{\theta_2} \left[ \ln \left( \frac{r_2}{R} \right) + i(\theta_1 + \theta_2) \right] Q(\phi) d\phi.$$ 

Therefore the stream function, given by $\psi = \text{Real} \{F(z)\}$, is

$$\psi(x,y) = UR \left[ \frac{2}{R} \int_{r_1}^{r_2} \theta_1 + \theta_2 \sigma(\phi) d\phi \right]$$

(25)

where $\theta_1$ and $\theta_2$ are functions of $x, y$ and $\phi$. Now, on a streamline $\psi$ is a constant and equation (25) can be solved to locate the position of the line. Here too a simple iteration procedure is used to find $y$ for a given $x$. Note that $\theta_1$ and $\theta_2$ are multi-valued functions and therefore a suitable branch-cut(s) should be selected to make them single-valued. A set of streamlines around the cylinders can be obtained by selecting suitable values for the stream function. The value corresponding to the separation streamlines may be obtained through equation (25) by considering any point on the front wetted surface. The streamline pattern for the case of $e = 1.5$ is plotted in Figure (9), where the angular shift of the near-wake is clearly indicated.
CONCLUSION

A potential flow surface source model is developed for two-dimensional time-averaged flow past two identical circular cylinders. The governing integral equation for the source strength distribution is solved numerically to obtain a complete solution. An approximate analytical method is presented too, which shows close agreement with the numerical results, except for cylinders with a very small gap.

The main advantages of the present model are that it is simpler to use and it does not require any empirical parameters. The model predicts the positions of the separating points and the corresponding back pressure coefficients. The results show that the effects of the presence of the adjacent body are to shift the separating points, resulting an angular shift of the near wake, and to create a non-uniform, higher wake under-pressure. These effects give rise to higher drag force on the body and generate a lift force.

The model can be extended readily for the general two-body problem of different body sizes with any flow direction respect to the bodies, if there is no direct interaction of the down-stream body with the wake of the up-stream one. In such situations there will be two integral equations for the source strength distributions on the two bodies with four unknown angular positions of the separation points. In principle, this method of analysis could be used for a general multi-body flow problem with any number of circular cylinders.

Finally, other possible use of the present model is worth mentioning. It is possible to calculate the effect of the presence of a plane solid surface or of blockage in a wind tunnel by using a system of images.
REFERENCES


